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Momentum-dependent contributions to the gravitational coupling of neutrinos in a medium

José F. Nieves
Laboratory of Theoretical Physics
Department of Physics, P.O. Box 23343
University of Puerto Rico, Río Piedras
Puerto Rico 00931-3343

Palash B. Pal
Theory Group, Saha Institute of Nuclear Physics,
1/AF Bidhan-Nagar, Calcutta 700064, India

Abstract

When neutrinos travel through a normal matter medium, the electron neutrinos couple differently to gravity compared to the other neutrinos, due to the presence of electrons in the medium and the absence of the other charged leptons. We calculate the momentum-dependent part of the matter-induced gravitational couplings of the neutrinos under such conditions, which arise at order g^2/M_W^4, and determine their contribution to the neutrino dispersion relation in the presence of a gravitational potential $\phi^{\rm ext}$. These new contributions vanish for the muon and tau neutrinos. For electron neutrinos with momentum K, they are of the order of the usual Wolfenstein term times the factor $(K^2/M_W^2)\phi^{\rm ext}$, for high energy neutrinos. In environments where the gravitational potential is substantial, such as those in the vicinity of Active Galactic Nuclei, they could be the dominant term in the neutrino dispersion relation. They must also be taken into account in the analysis of possible violations of the Equivalence Principle in the neutrino sector, in experimental settings involving high energy neutrinos traveling through a matter background.

1 Introduction

In the vacuum, a photon that propagates with momentum $\vec{K}$ in a constant gravitational potential $\phi^{\rm ext}$ has a dispersion relation given by

$$\omega_K = K(1 + 2\phi^{\rm ext})\,. \tag{1.1}$$

In fact, this formula holds not just for photons, but for any massless particle. In particular, it also holds for neutrinos in the limit in which their mass can be neglected. This result is a consequence of the universality of the gravitational interactions that is embodied in the Principle of Equivalence, according to which gravity couples to matter particles in a universal way; i.e., the graviton field couples to the stress-energy tensor of the particles with a coupling constant that is the same for all particle species.

Since the gravitational contribution to the neutrino dispersion relation is the same for all the neutrino flavors, it is not relevant in the context of neutrino oscillation experiments. That contribution yields a common factor in the evolution equation and therefore it drops out of the formulas for the flavor transition amplitudes.

On the other hand, it has been pointed out by Gasperini [1], and by Halprin and Leung [2], that a small violation of the equivalence principle, manifest as a difference in the couplings of the various neutrinos to gravity, would have observable consequences in various neutrino experiments. By the same token, precise measurements of various observable quantities in these experiments can place significant constraints on any deviations of the Equivalence Principle in the neutrino sector[3].

In our earlier work [4, 5] we considered neutrinos traveling through a background of normal matter, made of electrons and nucleons. By calculating the weak interaction corrections to the stress energy tensor of the neutrinos in the background, we showed that the lowest order gravitational couplings of the neutrinos are modified in a non-universal way. In summary, the magnitude of the background-induced contribution to the neutrino gravitational couplings was calculated to order g^2/M_W^2, the corrections to the neutrino dispersion relations in the presence of a gravitational potential were determined, and some of their possible phenomenological consequences were considered [6].

The corrections to the neutrino effective potential determined in Ref. [4] are independent of the neutrino momentum because, to order g^2/M_W^2, the weak interactions are local. Our purpose in the present work is to extend the calculation of the stress-energy tensor of the neutrinos, to include the order g^2/M_W^4 terms. These corrections arise by retaining the momentum-dependent terms of the W-propagator in the relevant one-loop diagrams for the neutrino gravitational vertex in matter. The procedure is analogous to that employed in Ref. [7, 8] to determine the non-local corrections to the neutrino effective potentials in matter in the absence of any external field.

The distinguishing feature of the $O(g^2/M_W^4)$ corrections that we determine in the present work is that they induce momentum-dependent terms in the effective potential of the neutrinos in the presence of a gravitational potential, which mimic the features of a non-universal gravitational coupling of the neutrinos at a fundamental level. We point out that our calculations, and the results based on them, do not depend on any physical assumption beyond those required by the standard model of particle interactions and the linearized theory of gravity, including the question of whether or not the neutrinos have a non-zero mass. Hence, the effects that we will consider are present at some level and it is conceivable that they are detectable in some favorable situations involving strong gravitational fields, such as those that exist in the vicinity of active galactic nuclei. In any case, these corrections in principle must be taken into account in the analysis of the tests

and constraints of possible violations of the Equivalence Principle in the neutrino sector, related to experimental observations involving neutrinos traveling through a background, as opposed to the vacuum.

The paper is organized as follows. In Sec. 2, we present the diagrams to be calculated and the Feynman rules needed for their calculation. We write down the expressions for vertex diagrams with internal Z-boson and W-boson lines in Sec. 3.1 and Sec. 3.2 respectively. This sets up the stage for Sec. 4, which contains most of the results of this paper. In this section, we start by setting up the general formalism which relates the neutrino gravitational vertex function to the neutrino self-energy in an external gravitational field. Then, in Sec. 4.2, we show that although the general vertex can depend on the unphysical parameter ξ appearing in the W and Z propagators, the dispersion relation is independent of it. The diagrams with internal Z-bosons lines do not contribute to the dispersion relations at $O(g^2/M_W^4)$. The $O(g^2/M_W^4)$ contributions of the different W-mediated diagrams to the self energy are calculated in Sec. 4.3, and they are collected together in Sec. 5 to find the neutrino dispersion relation to $O(g^2/M_W^4)$. We conclude with some comments in Section 6.

2 Feynman rules

The diagrams that we must calculate in this paper are shown in Figs. 1 and 2. In principle, we must consider also the diagrams in which any number of internal vector bosons is replaced by the corresponding unphysical Higgs bosons. In general, only by considering them we can attempt to prove that their contribution cancels the one that arises from the gauge-dependent piece of the gauge boson propagator. However, for simplicity, we will carry out the entire calculation in the limit $m_e \to 0$, in which case the diagrams involving the unphysical Higgs particles do not contribute. Hence, we do not consider them any further. Our task is therefore twofold: to calculate the $O(1/M_B^4)$ contribution terms of the diagrams shown in Figs. 1 and 2 where B is either the W or the Z-boson, and to show that the contribution from the gauge-dependent terms of the vector boson propagators are zero in the limit $m_e \to 0$.

The propagators and vertices that we need to perform the computation are as follows. The thermal propagator for a fermion of mass m_f is given by

$$iS_f(p) = (\not p + m_f)\left[\frac{i}{p^2 - m_f^2 + i\epsilon} - 2\pi\delta(p^2 - m_f^2)\eta_f(p)\right], \qquad (2.1)$$

where

$$\eta_f(p) = \frac{\theta(p\cdot v)}{e^{\beta(p\cdot v - \mu_f)} + 1} + \frac{\theta(-p\cdot v)}{e^{-\beta(p\cdot v - \mu_f)} + 1} \qquad (2.2)$$

with β being the inverse temperature, μ_f the chemical potential and v^μ the velocity four-vector of the medium.

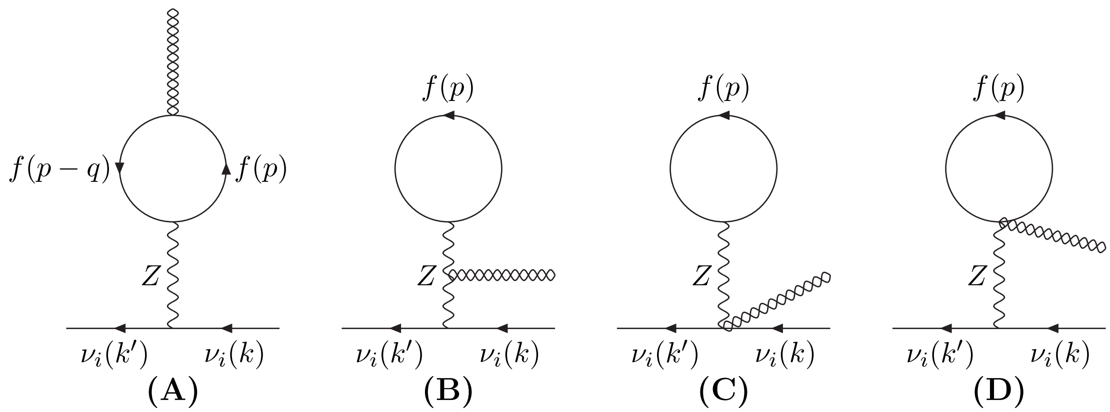


Figure 1: Z-exchange diagrams for the one-loop contribution to the gravitational vertex of any neutrino flavor ($i = e, \mu, \tau$) in a background of electrons and nucleons. The braided line represents the graviton.

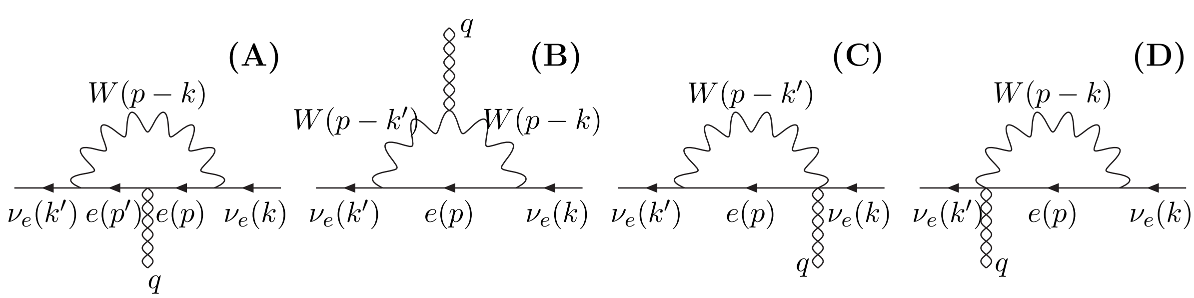


Figure 2: W-exchange diagrams for the one-loop contribution to the ν_e gravitational vertex in a background of electrons.

The propagators for a gauge boson $B = W, Z$ are given in the ξ gauge as

$$\Delta_{B\mu\nu}(q) = \frac{-1}{q^2 - M_B^2}\left\{\eta_{\mu\nu} - \frac{(1-1/\xi)q_\mu q_\nu}{q^2 - M_B^2/\xi}\right\}. \tag{2.3}$$

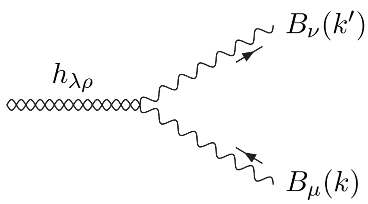
The contributions of order $1/M_B^2$ to the neutrino effective gravitational vertex were calculated in Refs. [4, 5]. Here we are interested in the terms next to the leading ones, of order $1/M_B^4$. For that purpose, it is sufficient to approximate the boson propagators that appear in the loop integrals by

$$\Delta_{B\mu\nu}(q) = \frac{\eta_{\mu\nu}}{M_B^2} + \Delta^{(4)}_{B\mu\nu}(q) + \Delta^{(\xi)}_{B\mu\nu}(q)\,, \tag{2.4}$$

where

$$\begin{aligned}
\Delta^{(4)}_{B\mu\nu}(q) &= \frac{1}{M_B^4}\left(q^2\eta_{\mu\nu} - q_\mu q_\nu\right)\,, \\
\Delta^{(\xi)}_{B\mu\nu}(q) &= \frac{\xi q_\mu q_\nu}{M_B^4}\,.
\end{aligned} \tag{2.5}$$

The remaining terms would yield contributions of order $1/M_B^6$ to the effective vertex, which we are not considering.



$$= i\kappa M_B^2 a'_{\lambda\rho\mu\nu} - i\kappa\left(C_{\lambda\rho\mu\nu}(k,k') + C^{(\xi)}_{\lambda\rho\mu\nu}(k,k')\right)$$

Figure 3: Feynman rule for the gravitational vertex coupling of a gauge boson $B = W, Z$ with the graviton.

We also need the Feynman rules for the graviton couplings of the vector bosons, including the dominant sub-leading terms that were not considered in our earlier work, as well as the gauge-dependent terms. To obtain them we need to start from the free Lagrangian, which for the W-bosons is

$$\mathscr{L}_g^{(W)} = \sqrt{-\mathsf{g}}\left\{-\frac{1}{2}g^{\mu\alpha}g^{\nu\beta}W^*_{\mu\nu}W_{\alpha\beta} + M_W^2 g^{\mu\nu}W^*_\mu W_\nu - \xi g^{\mu\nu}g^{\alpha\beta}(\partial_\mu W^*_\nu)(\partial_\alpha W^*_\beta)\right\}\,, \tag{2.6}$$

where $\mathsf{g} = \det(g_{\mu\nu})$. We now write

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}\,, \tag{2.7}$$

where κ is related to the Newton's constant by

$$\kappa = \sqrt{8\pi G}\,, \tag{2.8}$$

and $h_{\mu\nu}$ is the graviton field. Keeping up to first order terms in κ, we obtain the WW-graviton coupling in the form

$$\mathscr{L}_h^{(WW)} = -\kappa h^{\lambda\rho}(x)\left(\widehat{T}_{\lambda\rho}(x) + \widehat{T}^{(\xi)}_{\lambda\rho}(x)\right)\,, \tag{2.9}$$

where

$$\begin{aligned}
\widehat{T}_{\lambda\rho}(x) &= \frac{1}{2}\eta_{\lambda\rho}W^*_{\mu\nu}W^{\mu\nu} + 2W^*_{\lambda\mu}W^{\mu}{}_{\rho} - M_W^2 a'_{\lambda\rho\mu\nu}W^{*\nu}W^{\mu} \\
\widehat{T}^{(\xi)}_{\lambda\rho}(x) &= \xi\left\{\eta_{\lambda\rho}(\partial\cdot W^*)(\partial\cdot W) - (\partial_\lambda W^*_\rho + \partial_\rho W^*_\lambda)\partial\cdot W - \partial\cdot W^*(\partial_\lambda W_\rho + \partial_\rho W_\lambda)\right\}\,,
\end{aligned} \tag{2.10}$$

using the shorthand

$$a'_{\lambda\rho\mu\nu} = \eta_{\lambda\rho}\eta_{\mu\nu} - (\eta_{\lambda\mu}\eta_{\rho\nu} + \eta_{\lambda\nu}\eta_{\rho\mu})\,. \tag{2.11}$$

The Feynman rule for the WW-graviton coupling can then be written as in Fig. 3, with $a'_{\lambda\rho\mu\nu}$ as in Eq. (2.11), and

$$\begin{aligned}
C_{\lambda\rho\mu\nu}(k,k') &= \eta_{\lambda\rho}(\eta_{\mu\nu}k\cdot k' - k_\mu k'_\nu) - \eta_{\mu\nu}(k_\lambda k'_\rho + k'_\lambda k_\rho) \\
&\quad + k_\nu(\eta_{\lambda\mu}k'_\rho + \eta_{\rho\mu}k'_\lambda) + k'_\mu(\eta_{\lambda\nu}k_\rho + \eta_{\rho\nu}k_\lambda) \\
&\quad - k\cdot k'(\eta_{\lambda\mu}\eta_{\rho\nu} + \eta_{\lambda\nu}\eta_{\rho\mu}) \\
C^{(\xi)}_{\lambda\rho\mu\nu}(k,k') &= \xi\left\{\eta_{\lambda\rho}k_\mu k'_\nu - k'_\lambda k_\mu\eta_{\rho\nu} - k'_\rho k_\mu\eta_{\lambda\nu} - k_\lambda k'_\nu\eta_{\rho\mu} - k_\rho k'_\nu\eta_{\lambda\mu}\right\}\,.
\end{aligned} \tag{2.12}$$

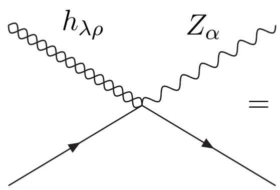
The rule for the couplings of the Z boson are the same.

The other couplings necessary for the present calculation have been deduced in Ref. [4]. For the gravitational coupling of a fermion of mass m_f, the Feynman rule involves the factor $-i\kappa V_{\lambda\rho}(p,p')$, with

$$V_{\lambda\rho}(p,p') = \frac{1}{4}\Big[\gamma_\lambda(p+p')_\rho + \gamma_\rho(p+p')_\lambda\Big] - \frac{1}{2}\Big[\not{p} + \not{p}' - 2m_f\Big]\,, \tag{2.13}$$

where p and p' denote the momentum of the incoming and the outgoing fermion, respectively. There is also a fermion vertex involving both a graviton and a gauge boson line, and its Feynman rule is given in Fig. 4, with the definition

$$a_{\lambda\rho\alpha\beta} = \eta_{\lambda\rho}\eta_{\alpha\beta} - \frac{1}{2}(\eta_{\lambda\alpha}\eta_{\rho\beta} + \eta_{\rho\alpha}\eta_{\lambda\beta})\,. \tag{2.14}$$



$$= -i\tfrac{g}{2\cos\theta_W}\kappa a_{\lambda\rho\alpha\beta}\gamma^\beta(X_f + Y_f\gamma_5)$$

Figure 4: Feynman rule for the graviton coupling involving the Z-boson and a fermion. $a_{\lambda\rho\alpha\beta}$ has been defined in Eq. (2.14) and, in the standard model, $X_\nu = -Y_\nu = 1/2$ for a neutrino, and $-X_e = X_p = \frac{1}{2} - 2\sin^2\theta_W$, $X_n = -\frac{1}{2}$, $Y_e = \frac{1}{2}$, $Y_n = -Y_p = \frac{1}{2}g_A$, with $g_A = 1.26$ being the renormalization constant of the axial-vector current of the nucleon. The analogous couplings involving the W boson are obtained by replacing $\frac{g}{2\cos\theta_W}$ by $\frac{g}{\sqrt{2}}$, X_f by $\frac{1}{2}$ and Y_f by $-\frac{1}{2}$.

3 Calculation of the vertex function diagrams

3.1 The Z-mediated diagrams

The diagrams, which are shown in Fig. 1, are the same as in Ref. [4]. We use the same notation and conventions given in that reference for the weak-neutral-current couplings in the Lagrangian [e.g., Eqs. (2.31)-(2.35) there], summarized in Fig. 4 here. The Z-propagator is external to the loop, so all the algebraic manipulations of the loop integral are performed exactly as in Ref. [4]. As in that reference, we define

$$b^{(Z)} = \sum_f X_f b_f \,, \tag{3.1}$$

where the sum is over the fermions species in the background, and[1]

$$\begin{aligned} b_f &= 4\sqrt{2}G_F \int \frac{d^4p}{(2\pi)^3}\delta(p^2 - m_f^2)\eta_f(p)\, p\cdot v \\ &= \sqrt{2}G_F(n_f - n_{\bar{f}})\,, \end{aligned} \tag{3.2}$$

with $n_{f,\bar{f}}$ being the number densities of the particles and antiparticles respectively. We denote by k, k' the momentum of the initial and final neutrino respectively, and the momentum of the outgoing graviton is

$$q = k - k' \,. \tag{3.3}$$

For Fig. 1A we then obtain

$$\Gamma'^{(1A)}_{\lambda\rho} = \left(\gamma^\alpha + \frac{q^2\gamma^\alpha - (1-\xi)q^\alpha \not{q}}{M_Z^2}\right)\left(\Lambda^{(Z)}_{\lambda\rho\alpha} - b^{(Z)}\eta_{\lambda\rho}v_\alpha\right) L \,, \tag{3.4}$$

[1] We take the opportunity to correct a typographical error in Eq. (3.20) of Ref. [4]

where

$$
\begin{aligned}
\Lambda^{(Z)}_{\lambda\rho\alpha} &= \frac{g_Z^2}{M_Z^2}\sum_f \int \frac{d^3p}{2E_f(2\pi)^3}\left\{X_f(f_f - f_{\overline{f}})\left[\frac{N^{(1)}_{\lambda\rho\alpha}(p_f,q)}{q^2-2p_f\cdot q} + (q\to -q)\right]\right. \\
&\quad \left. - Y_f(f_f + f_{\overline{f}})\left[\frac{N^{(2)}_{\lambda\rho\alpha}(p_f,q)}{q^2-2p_f\cdot q} - (q\to -q)\right]\right\}, \qquad (3.5)
\end{aligned}
$$

with f_f and $f_{\overline{f}}$ denoting the momentum distribution functions for particles and antiparticles and

$$
\begin{aligned}
N^{(1)}_{\lambda\rho\alpha}(p,q) &\equiv (2p-q)_\lambda[2p_\rho p_\alpha - (p_\alpha q_\rho + q_\alpha p_\rho) + (p\cdot q)\eta_{\alpha\rho}] + (\lambda \leftrightarrow \rho) \\
N^{(2)}_{\lambda\rho\alpha}(p,q) &\equiv (2p-q)_\lambda i\epsilon_{\rho\alpha\beta\sigma}q^\beta p^\sigma + (\lambda\leftrightarrow\rho)\,. \qquad (3.6)
\end{aligned}
$$

In Fig. 1B, the Z line attached to the fermion loop carries zero momentum, in which case the $C_{\lambda\rho\mu\nu}$ and $C^{(\xi)}_{\lambda\rho\mu\nu}$ terms of the ZZ graviton couplings do not contribute. Hence the contribution from this diagram is given by

$$
\Gamma^{(1B)}_{\lambda\rho} = -b^{(Z)}a'_{\lambda\rho\alpha\beta}\left(\gamma^\alpha + \frac{q^2\gamma^\alpha - (1-\xi)q^\alpha \not{q}}{M_Z^2}\right)Lv^\beta\,, \qquad (3.7)
$$

where $a'_{\lambda\rho\alpha\beta}$ has been defined in Eq. (2.11).

In Fig. 1C, the Z-boson line has zero momentum flowing through it. As a consequence, the $O(M_Z^{-4})$ terms of the Z-propagator do not contribute and the amplitude for this diagram is the same as obtained in Refs. [4, 5], viz.,

$$
\Gamma^{(1C)}_{\lambda\rho} = b^{(Z)}a_{\lambda\rho\alpha\beta}\gamma^\alpha L v^\beta\,, \qquad (3.8)
$$

where $a_{\lambda\rho\alpha\beta}$ was defined in Eq. (2.14). Finally, for Fig. 1D, we get

$$
\Gamma^{(1D)}_{\lambda\rho} = b^{(Z)}a_{\lambda\rho\alpha\beta}\left(\gamma^\alpha + \frac{q^2\gamma^\alpha - (1-\xi)q^\alpha\not{q}}{M_Z^2}\right)Lv^\beta\,. \qquad (3.9)
$$

In this way, summing up all the diagrams of Fig. 1, we obtain the total contribution from the Z-diagrams to the effective vertex

$$
\begin{aligned}
\Gamma'^{(1)}_{\lambda\rho} &= \Lambda^{(Z)}_{\mu\nu\alpha}\gamma^\alpha L \\
&\quad + \left(\frac{q^2\gamma^\alpha - (1-\xi)q^\alpha\not{q}}{M_Z^2}\right)L\left(\Lambda^{(Z)}_{\mu\nu\alpha} + b^{(Z)}\Big[\eta_{\lambda\rho}\eta_{\alpha\beta} + \frac{1}{2}(\eta_{\lambda\alpha}\eta_{\rho\beta} + \eta_{\rho\alpha}\eta_{\lambda\beta})\Big]v^\beta\right), \qquad (3.10)
\end{aligned}
$$

where the first term on the right hand side represents the leading order contribution, of order $(1/M_Z^2)$, whose detailed evaluation was the subject of Ref. [4]. It was shown in that reference that,

for a static and constant gravitational potential, which involves taking the $q \to 0$ of the vertex function in the proper way, the contribution from the $\Lambda^{(Z)}_{\mu\nu\alpha}$ term to the neutrino self-energy is a well defined quantity. Borrowing that result here, it follows that the remaining terms in Eq. (3.10), which involve the same quantity $\Lambda^{(Z)}_{\mu\nu\alpha}$, yield a contribution to the neutrino self-energy that depends on the derivatives of the gravitational field. Therefore, in the presence of a static and homogeneous potential, the $O(1/M_Z^4)$ contributions to the neutrino self-energy vanish. Since these are the only diagrams that contribute to the $\nu_{\mu,\tau}$ dispersion relation if there is no μ or τ in the background, as we assume, only the dispersion relation of the ν_e is modified by the $O(1/M_B^4)$ terms. These arise from the W diagrams that we consider next.

3.2 The W-mediated diagrams

We now discuss the diagrams involving the W-bosons, shown in Fig. 2, which are relevant only for the electron neutrino. In what follows, unless we explicitly mention it otherwise, we are referring only to this type of neutrino.

3.2.1 Diagram 2A

Let us start with Fig. 2A. It gives

$$\Gamma^{(2A)}_{\lambda\rho} = -\frac{ig^2}{2}\int \frac{d^4p}{(2\pi)^4}\gamma^\alpha L iS_e(p')V_{\lambda\rho}(p,p')iS_e(p)\gamma^\beta L\Delta_{W\alpha\beta}(p-k)\,, \tag{3.11}$$

where

$$p' = p - q\,. \tag{3.12}$$

By substituting the W propagator in Eq. (3.11) three terms are produced. The $O(1/M_W^2)$ term has already been calculated in Ref. [4]. Separating it out, we can write the dispersive part of $\Gamma'^{(2A)}_{\lambda\rho}$ in the form

$$\Gamma'^{(2A)}_{\lambda\rho} = \Lambda^{(W)}_{\lambda\rho} - b_e \not{k} L\eta_{\lambda\rho} + G^{(A)}_{\lambda\rho} + H^{(A)}_{\lambda\rho}\,, \tag{3.13}$$

where $G^{(A)}_{\lambda\rho}$, which contains all the $O(1/M_W^4)$ contributions that are ξ-independent, is given by

$$\begin{aligned} G^{(A)}_{\lambda\rho} &= -\frac{g^2}{2}\int \frac{d^4p}{(2\pi)^3}\gamma^\alpha L\not{p}'V_{\lambda\rho}(p,p')\not{p}\gamma^\beta L\,\Delta^{(4)}_{W\alpha\beta}(p-k) \\ &\times \left(\frac{\delta(p^2)\eta_e(p)}{p'^2} + \frac{\delta(p'^2)\eta_e(p')}{p^2}\right)\,, \end{aligned} \tag{3.14}$$

while $H^{(A)}_{\lambda\rho}$ is given by the same expression as Eq. (3.14) but with the replacement

$$\Delta^{(4)}_{W\alpha\beta} \to \Delta^{(\xi)}_{W\alpha\beta}\,. \tag{3.15}$$

3.2.2 Diagram 2B

Proceeding in similar fashion for this diagram, we borrow the leading order result derived in Ref. [4] and write

$$\Gamma'^{(2B)}_{\lambda\rho} = b_e\Big[-\eta_{\lambda\rho}\not{p} - \frac{1}{2}(\gamma_\lambda\not{p}\gamma_\rho + \gamma_\rho\not{p}\gamma_\lambda)\Big]L + G^{(B)}_{\lambda\rho} + H^{(B)}_{\lambda\rho}\,, \tag{3.16}$$

where $G^{(B)}_{\lambda\rho}$ and $H^{(B)}_{\lambda\rho}$ contain the non-leading terms, separated according to whether they are ξ-independent or not, respectively. These non-leading terms can come either from the non-leading contribution in the WW-graviton coupling or in the W-propagator. Accordingly, we get the following terms

$$\begin{aligned}
G^{(B)}_{\lambda\rho} &= \frac{g^2}{2}\int\frac{d^4p}{(2\pi)^3}\delta(p^2)\eta_e(p)\gamma^\alpha L\not{p}\gamma^\beta L \\
&\times \left[-\frac{1}{M_W^4}C_{\lambda\rho\alpha\beta}(p-k',p-k) + \eta^{\mu\nu}a'_{\lambda\rho\alpha\mu}\Delta^{(4)}_{W\nu\beta}(p-k) + \eta^{\mu\nu}a'_{\lambda\rho\mu\beta}\Delta^{(4)}_{W\alpha\nu}(p-k')\right] \\
H^{(B)}_{\lambda\rho} &= \frac{g^2}{2}\int\frac{d^4p}{(2\pi)^3}\delta(p^2)\eta_e(p)\gamma^\alpha L\not{p}\gamma^\beta L \\
&\times \left[-\frac{1}{M_W^4}C^{(\xi)}_{\lambda\rho\alpha\beta}(p-k',p-k) + \eta^{\mu\nu}a'_{\lambda\rho\alpha\mu}\Delta^{(\xi)}_{W\nu\beta}(p-k) + \eta^{\mu\nu}a'_{\lambda\rho\mu\beta}\Delta^{(\xi)}_{W\alpha\nu}(p-k')\right]
\end{aligned} \tag{3.17}$$

3.2.3 Diagrams 2C and D

Similarly, we define the non-leading contributions by writing

$$\Gamma^{(2C)}_{\lambda\rho} = b_e\Big[\eta_{\lambda\rho}\not{p} + \frac{1}{4}(\gamma_\lambda\not{p}\gamma_\rho + \gamma_\rho\not{p}\gamma_\lambda)\Big]L + G^{(C)}_{\lambda\rho} + H^{(C)}_{\lambda\rho}\,, \tag{3.18}$$

$$\Gamma^{(2D)}_{\lambda\rho} = b_e\Big[\eta_{\lambda\rho}\not{p} + \frac{1}{4}(\gamma_\lambda\not{p}\gamma_\rho + \gamma_\rho\not{p}\gamma_\lambda)\Big]L + G^{(D)}_{\lambda\rho} + H^{(D)}_{\lambda\rho}\,. \tag{3.19}$$

For these diagrams, the non-leading terms come only from the W propagator. Thus we obtain

$$G^{(C)}_{\lambda\rho} = -\frac{g^2}{2}a_{\lambda\rho\beta\mu}\int\frac{d^4p}{(2\pi)^3}\delta(p^2)\eta_e(p)\gamma_\alpha L\not{p}\gamma^\beta L\Delta^{(4)\mu\alpha}_W(p-k') \tag{3.20}$$

$$G^{(D)}_{\lambda\rho} = -\frac{g^2}{2}a_{\lambda\rho\alpha\mu}\int\frac{d^4p}{(2\pi)^3}\delta(p^2)\eta_e(p)\gamma^\alpha L\not{p}\gamma_\beta L\Delta^{(4)\beta\mu}_W(p-k) \tag{3.21}$$

The corresponding formulas for $H^{(C,D)}_{\lambda\rho}$ are given by the same expressions, with the replacement indicated in Eq. (3.15).

4 Self-energy in a constant potential

4.1 General formalism

In the presence of a static and homogeneous gravitational potential, the neutrino self-energy, including the $O(g^2/M_W^4)$ corrections, is given by

$$\Sigma = \Sigma_{\rm mat} + \Sigma_G^{(2)} + \Sigma_G^{(4)} + \Sigma_G^{(\xi)} \tag{4.1}$$

where $\Sigma_{\rm mat}$ stands for the purely weak self-energy terms which do not depend on the gravitational potential, including the $O(g^2/M_W^4)$ corrections, while $\Sigma_G^{(2)}$ is the quantity calculated in Ref. [4] [defined in Eq. (4.8) there]. In analogy with the expression given in Eq. (4.9) of that reference, we can write here

$$\begin{aligned}
\Sigma_G^{(4)}(\omega, \vec{K}) &= \phi^{\rm ext}(2v^\lambda v^\rho - \eta^{\lambda\rho})G_{\lambda\rho}(0, \vec{\mathcal{Q}} \to 0)\,, \\
\Sigma_G^{(\xi)}(\omega, \vec{K}) &= \phi^{\rm ext}(2v^\lambda v^\rho - \eta^{\lambda\rho})H_{\lambda\rho}(0, \vec{\mathcal{Q}} \to 0)\,,
\end{aligned} \tag{4.2}$$

where $G_{\lambda\rho}(\Omega, \vec{\mathcal{Q}})$ and $H_{\lambda\rho}(\Omega, \vec{\mathcal{Q}})$ denote the sum of the various gauge-independent and gauge-dependent terms $G_{\lambda\rho}^{(X)}$ and $H_{\lambda\rho}^{(X)}$ respectively, whose integral expressions have been given in Eqs. (3.14), (3.17), (3.20) and (3.21). As we already argued at the end of Sec. 3.1, the $O(g^2/M_Z^4)$ terms from the Z-mediated diagrams do not contribute to the self-energy of the neutrinos, and we have to calculate the contributions only from the W-mediated diagrams in Eq. (4.2). In Eq. (4.2) we have indicated the dependence of the various quantities[2] on the kinematic variables of the problem. These variables are defined by writing

$$\begin{aligned}
k^\mu &= (\omega, \vec{K}) \\
q^\mu &= (\Omega, \vec{\mathcal{Q}})
\end{aligned} \tag{4.3}$$

in the rest frame of the medium where

$$v^\mu = (1, \vec{0})\,. \tag{4.4}$$

In what follows, we indicate the dependence of the various functions on these variables when it is necessary, but we omit it otherwise.

The chirality of the neutrino interactions dictate that Σ has the decomposition

$$\Sigma = (a\not{k} + b\not{v})L\,, \tag{4.5}$$

where a and b are in general functions of ω and K. It is convenient to split a and b as

$$\begin{aligned}
a &= a_{\rm mat} + a_G^{(2)} + a_G^{(4)} + a_G^{(\xi)} \\
b &= b_{\rm mat} + b_G^{(2)} + b_G^{(4)} + b_G^{(\xi)}\,,
\end{aligned} \tag{4.6}$$

[2] Although we do not show it explicitly, $G^{(X)}$ depends on ω and K as well.

corresponding to the decomposition given in Eq. (4.1). Eq. (4.5) implies that $a_G^{(4)}$ and $b_G^{(4)}$ can be determined by means of the formulas

$$\begin{aligned} a_G^{(4)} &= \frac{\omega A^{(4)} - B^{(4)}}{K^2} \\ b_G^{(4)} &= \frac{\omega B^{(4)} - k^2 A^{(4)}}{K^2}\,, \end{aligned} \tag{4.7}$$

and similarly

$$\begin{aligned} a_G^{(\xi)} &= \frac{\omega A_\xi - B_\xi}{K^2} \\ b_G^{(\xi)} &= \frac{\omega B_\xi - k^2 A_\xi}{K^2}\,, \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} B^{(4)}(\omega, K) &= \phi^{\rm ext}(2v^\lambda v^\rho - \eta^{\lambda\rho})\frac{1}{2}{\rm Tr}\not{k}G_{\lambda\rho}(0, \mathcal{Q} \to 0)\,, \\ A^{(4)}(\omega, K) &= \phi^{\rm ext}(2v^\lambda v^\rho - \eta^{\lambda\rho})\frac{1}{2}{\rm Tr}\not{v}G_{\lambda\rho}(0, \mathcal{Q} \to 0)\,, \end{aligned} \tag{4.9}$$

and similarly

$$\begin{aligned} B_\xi(\omega, K) &= \phi^{\rm ext}(2v^\lambda v^\rho - \eta^{\lambda\rho})\frac{1}{2}{\rm Tr}\not{k}H_{\lambda\rho}(0, \mathcal{Q} \to 0)\,, \\ A_\xi(\omega, K) &= \phi^{\rm ext}(2v^\lambda v^\rho - \eta^{\lambda\rho})\frac{1}{2}{\rm Tr}\not{v}H_{\lambda\rho}(0, \mathcal{Q} \to 0)\,. \end{aligned} \tag{4.10}$$

As discussed in Ref. [4], the dispersion relations are obtained by solving

$$\begin{aligned} \omega_K &= K + \frac{b(\omega_K, K)}{1 - a(\omega_K, K)} \\ \bar{\omega}_K &= K - \frac{b(-\bar{\omega}_K, K)}{1 - a(-\bar{\omega}_K, K)} \end{aligned} \tag{4.11}$$

for the neutrinos and antineutrinos, respectively. In the context of our perturbative approach, the solutions are given approximately by

$$\begin{aligned} \omega_K &= K + [1 + a(K, K)]b(K, K) \\ \bar{\omega}_K &= K - [1 + a(-K, K)]b(-K, K)\,. \end{aligned} \tag{4.12}$$

Using the result for the dispersion relation obtained in Ref. [4] [given in Eq. (4.32) of that reference], it is easy to see from Eq. (4.7) that in the present case

$$\begin{aligned} \omega_K &= K + 2\phi^{\rm ext}K + (1 + \phi^{\rm ext})b_{\rm mat} + b_G^{(2)} + \frac{B^{(4)}(K, K)}{K} + \frac{B_\xi(K, K)}{K} \\ \bar{\omega}_K &= K + 2\phi^{\rm ext}K + (1 + \phi^{\rm ext})\bar{b}_{\rm mat} - b_G^{(2)} + \frac{B^{(4)}(-K, K)}{K} + \frac{B_\xi(-K, K)}{K}\,. \end{aligned} \tag{4.13}$$

Here $b_G^{(2)}$ is the contribution calculated in Ref. [4] [denoted by simply b_G and given by the formula in Eq. (4.30) there], while $b_{\rm mat}$ and $\bar{b}_{\rm mat}$ are the usual contributions independent of the gravitational potential, for neutrinos and antineutrinos respectively, including the $O(g^2/M_W^4)$ terms. These can be obtained from the references cited[7, 8], and we will quote explicit formulas for our cases of interest in Sec. 5. Thus, what remains is simply to calculate the coefficient $B^{(4)}$ by means of Eq. (4.9), using the integral formulas for the various functions $G_{\lambda\rho}^{(X)}$ obtained in the previous section. However, before we proceed to do that, we will now show that the quantity $B_\xi(\pm K, K)$, which is calculated in similar fashion, is in fact zero so that we finally obtain the gauge invariant result

$$\begin{aligned}
\omega_K &= K + 2\phi^{\rm ext}K + (1+\phi^{\rm ext})b_{\rm mat} + b_G^{(2)} + \frac{B^{(4)}(K,K)}{K} \\
\bar{\omega}_K &= K + 2\phi^{\rm ext}K + (1+\phi^{\rm ext})\bar{b}_{\rm mat} - b_G^{(2)} + \frac{B^{(4)}(-K,K)}{K} \,.
\end{aligned} \tag{4.14}$$

4.2 Gauge independence of dispersion relations

We calculate B_ξ using Eq. (4.10). We denote by $B_\xi^{(X)}$ the contribution from each term $H_{\lambda\rho}^{(X)}$ that arises in that formula. Carrying out the trace operation first, a straightforward calculation yields the expressions

$$\begin{aligned}
B_\xi^{(A)}(\omega, K) &= \phi^{\rm ext}\left(-\frac{g^2\xi}{2M_W^4}\right) k^2(2v^\lambda v^\rho - \eta^{\lambda\rho})\left[X_{\lambda\rho}^{(1)} + Y_{\lambda\rho}(0, \mathcal{Q}\to 0)\right] \\
B_\xi^{(B)}(\omega, K) &= \phi^{\rm ext}\left(-\frac{g^2\xi}{2M_W^4}\right) k^2(2v^\lambda v^\rho - \eta^{\lambda\rho})\left[a'_{\lambda\rho\alpha\beta}X^{(2)\alpha\beta} + X_{\lambda\rho}^{(3)}\right] \\
B_\xi^{(C)}(\omega, K) + B_\xi^{(D)}(\omega, K) &= \phi^{\rm ext}\left(\frac{g^2\xi}{2M_W^4}\right) k^2(2v^\lambda v^\rho - \eta^{\lambda\rho})a_{\lambda\rho\alpha\beta}X^{(2)\alpha\beta} \,,
\end{aligned} \tag{4.15}$$

where

$$\begin{aligned}
X_{\lambda\rho}^{(1)} &= -\int \frac{d^4p}{(2\pi)^3}\delta(p^2)\eta_e(p)\left[\frac{1}{2}\eta_{\lambda\rho}(k\cdot p) + 2p_\lambda p_\rho\right] \\
X_{\alpha\beta}^{(2)} &= \int \frac{d^4p}{(2\pi)^3}\delta(p^2)\eta_e(p)\left[2p_\alpha p_\beta - p_\alpha k_\beta - p_\beta k_\alpha\right] \\
X_{\lambda\rho}^{(3)} &= \int \frac{d^4p}{(2\pi)^3}\delta(p^2)\eta_e(p)\left[\eta_{\lambda\rho}p\cdot k + 4p_\lambda p_\rho - 2p_\lambda k_\rho - 2p_\rho k_\lambda\right] \\
Y_{\lambda\rho}(\Omega, \mathcal{Q}) &= \int \frac{d^4p}{(2\pi)^3}\delta(p^2)\eta_e(p)\left[\frac{j_{\lambda\rho}}{q^2 - 2p\cdot q} + (q\to -q)\right] \,,
\end{aligned} \tag{4.16}$$

with

$$j_{\lambda\rho} = \frac{1}{4}(2p-q)_\lambda[(2k\cdot p - q\cdot k)p_\rho - (k\cdot p)q_\rho + (p\cdot q)k_\rho] + (\lambda \leftrightarrow \rho)\,. \tag{4.17}$$

The terms $X^{(1,2)}_{\lambda\rho}$ can be expressed very simply in terms of integrals over the electron distribution functions while $Y_{\lambda\rho}(0, \mathcal{Q} \to 0)$ can be evaluated following the techniques used to evaluate similar integrals in Ref. [4]. However, for our purposes here, the explicit results for them are not needed. For consistency, we just note that, despite appearances, $(2v^\lambda v^\rho - \eta^{\lambda\rho})Y_{\lambda\rho}(0, \mathcal{Q} \to 0)$ is well-defined (see Appendix A). Since each expression in Eq. (4.15) contains an explicit factor of k^2,we then obtain

$$B_\xi(\pm K, K) = 0\,, \tag{4.18}$$

which completes the proof of Eq. (4.14).

4.3 Calculation of $B^{(4)}$

As a byproduct of the gauge invariance proof given above, it follows that the longitudinal part of $\Delta^{(4)}_{W\mu\nu}$ does not contribute to the dispersion relations, which involve $B^{(4)}(\pm K, K)$. Therefore, to obtain the dispersion relations, we can use

$$k^2 = \omega^2 - K^2 = 0 \tag{4.19}$$

and substitute

$$\Delta^{(4)}_{W\mu\nu}(q) \to \eta_{\mu\nu}\frac{q^2}{M_W^4} \tag{4.20}$$

in the formulas for the $G^{(X)}_{\lambda\rho}$. Corresponding to each $G^{(X)}_{\lambda\rho}$, there is a term in $B^{(4)}$, and according to Eq. (4.9) we have

$$B^{(4)}(K, K) = \phi^{\rm ext}\left(\sum_{X=A,B,C,D} G^{(X)}(0, \mathcal{Q} \to 0)\right)\,, \tag{4.21}$$

where

$$G^{(X)}(\Omega, \mathcal{Q}) = (2v^\lambda v^\rho - \eta^{\lambda\rho})\frac{1}{2}\mathrm{Tr}\not{k}G^{(X)}_{\lambda\rho}\,. \tag{4.22}$$

We consider the computation of each of the $G^{(X)}(\Omega, \mathcal{Q})$ separately.

4.3.1 Computation of $G^{(A)}$

In the term containing $\eta_e(p')$ in the integral given in Eq. (3.14), we make the change of variables $p \to p + q$. Then using the replacement indicated in Eq. (4.20) for the W propagator, and remembering

that we are setting $m_e \to 0$, we obtain

$$\frac{1}{2}\mathrm{Tr}\not{k}G^{(A)}_{\lambda\rho} = -\frac{g^2}{2M_W^4}\int\frac{d^4p}{(2\pi)^3}\delta(p^2)\eta_e(p)\Bigg[\frac{(p-k)^2}{q^2-2p\cdot q}\Big(\mathscr{M}_{\lambda\rho}(q)-\mathscr{M}'_{\lambda\rho}(q)\Big) + \frac{(p-k')^2}{q^2+2p\cdot q}\Big(\mathscr{M}_{\lambda\rho}(-q)+\mathscr{M}'_{\lambda\rho}(-q)\Big)\Bigg], \tag{4.23}$$

where

$$\mathscr{M}_{\lambda\rho}(q) = -\frac{1}{2}\mathrm{Tr}\Big[\not{k}\not{p}V_{\lambda\rho}(p,p-q)(\not{p}-\not{q})\Big], \tag{4.24}$$

$$\mathscr{M}'_{\lambda\rho}(q) \equiv -\frac{1}{2}\mathrm{Tr}\Big[\not{k}\not{p}V_{\lambda\rho}(p,p-q)(\not{p}-\not{q})\gamma_5\Big]. \tag{4.25}$$

In writing these forms, we have used the cyclicity of trace and the contraction identity $\gamma_\alpha\not{k}\gamma^\alpha = -2\not{k}$. We have also used the identity $C\gamma_\mu C^{-1} = -\gamma_\mu^T$ to invert the order of the gamma matrices inside the trace in the second term in Eq. (4.23), as well as the property $V_{\lambda\rho}(p,p') = V_{\lambda\rho}(p',p)$. The trace operation can be carried out straightforwardly. The $\mathscr{M}'_{\lambda\rho}$ terms does not contribute to the final result. As for the $\mathscr{M}_{\lambda\rho}$ terms, we carry out the integral over p^0 with the help of the δ-function and substitute the resulting expression in Eq. (4.22). This gives

$$\begin{aligned} G^{(A)}(\Omega,\vec{\mathscr{Q}}) = & -\frac{g^2}{M_W^4}\int\frac{d^3p}{(2\pi)^3 2E}\Bigg[\frac{k^2-2k\cdot p}{q^2-2p\cdot q}\mathscr{M}(p,q)f_e + \frac{k^2+2k\cdot p}{q^2+2p\cdot q}\mathscr{M}(-p,q)f_{\bar{e}} \\ & +\frac{k'^2-2k'\cdot p}{q^2+2p\cdot q}\mathscr{M}(p,-q)f_e + \frac{k'^2+2k'\cdot p}{q^2-2p\cdot q}\mathscr{M}(-p,-q)f_{\bar{e}} \\ & -\frac{1}{2}k\cdot p\left[k^2+k'^2-2p\cdot(k+k')\right]f_e + \frac{1}{2}k\cdot p\left[k^2+k'^2+2p\cdot(k+k')\right]f_{\bar{e}}\Bigg], \end{aligned} \tag{4.26}$$

where

$$\begin{aligned} \mathscr{M}(p,q) = & \Big[-4(p\cdot v)^2+4p\cdot v\, q\cdot v-(q\cdot v)^2\Big]k\cdot p \\ & +\Big[-2p\cdot v+q\cdot v\Big]k\cdot v\, p\cdot q + \Big[2p\cdot v-q\cdot v\Big]p\cdot v\, k\cdot q\,. \end{aligned} \tag{4.27}$$

To evaluate $G^{(A)}(0,\vec{\mathscr{Q}}\to 0)$ we proceed as follows. We carry out the integration in the rest frame of the background. In that frame Eqs. (4.4) and (4.3) hold, and similarly we write

$$p^\mu = (\mathcal{E},\vec{\mathcal{P}}), \tag{4.28}$$

where $\mathcal{E} = |\vec{\mathcal{P}}|$ because of the δ-function that appears in Eq. (4.23). In addition we use the fact that k^μ satisfies Eq. (4.19). We next evaluate Eq. (4.26) in the static limit $\Omega = 0$ and take $\vec{\mathscr{Q}} \to 0$

afterwards. The terms without any denominator can be easily evaluated in this limit by putting $k = k'$. The results can be conveniently expressed by defining the integrals

$$I_{\mu_1 \mu_2 \cdots \mu_r} \equiv \int \frac{d^3\mathcal{P}}{(2\pi)^3 2\mathcal{E}}\, \mathcal{P}_{\mu_1} \mathcal{P}_{\mu_2} \cdots \mathcal{P}_{\mu_r} \Big[f_e(\mathcal{E}) + (-1)^r f_{\bar{e}}(\mathcal{E}) \Big] , \tag{4.29}$$

so that the contribution of these terms to $G^{(A)}(0, \vec{\mathcal{Q}} \to 0)$ can be written as

$$- \left(\frac{g^2}{M_W^4} \right) 2 k^\mu k^\nu I_{\mu\nu} . \tag{4.30}$$

It is now useful to introduce the scalar integrals, for any fermion f,

$$\begin{aligned} J_n^{(f)} &\equiv \int \frac{d^3\mathcal{P}}{(2\pi)^3 2\mathcal{E}} \mathcal{E}^n \Big[f_f(\mathcal{E}) + (-1)^n f_{\bar{f}}(\mathcal{E}) \Big] \\ &= \frac{1}{4\pi^2} \int d\mathcal{E}\; \mathcal{E}^{n+1} \Big[f_f(\mathcal{E}) + (-1)^n f_{\bar{f}}(\mathcal{E}) \Big] , \end{aligned} \tag{4.31}$$

for which we give explicit formulas in Sec. 5. Further simplification is then obtained by noting that the integrals in Eq. (4.29) can be determined in terms of the $J_n^{(e)}$; e.g.,

$$\begin{aligned} I_\mu &= J_1^{(e)} v_\mu , \\ I_{\mu\nu} &= \frac{1}{3} (-\eta_{\mu\nu} + 4 v_\mu v_\nu) J_2^{(e)} , \\ I_{\mu\nu\lambda} &= \left[-\frac{1}{3} \Big(\eta_{\mu\nu} v_\lambda + \eta_{\nu\lambda} v_\mu + \eta_{\lambda\mu} v_\nu \Big) + 2 v_\mu v_\nu v_\lambda \right] J_3^{(e)} . \end{aligned} \tag{4.32}$$

Thus the contribution of Eq. (4.30) reduces to

$$- \left(\frac{g^2}{M_W^4} \right) \frac{8}{3} \omega^2 J_2^{(e)} . \tag{4.33}$$

Despite appearances, the remaining terms in Eq. (4.26), which contain q in the denominator, yield a well-defined result in the $(\Omega = 0, \vec{\mathcal{Q}} \to 0)$ limit. We relegate the details of the calculation to Appendix B and give the final result here,

$$G^{(A)}(0, \vec{\mathcal{Q}} \to 0) = \frac{g^2}{M_W^4} \left[\frac{28}{3} \omega^2 J_2^{(e)} + 4 \omega J_3^{(e)} \right] , \tag{4.34}$$

including the contribution of the regular terms given in Eq. (4.33).

4.3.2 Computation of $G^{(B)}$

For $G^{(B)}$, we can directly put $q = 0$ and use Eq. (4.20) in Eq. (3.17). We then calculate

$$\begin{aligned} \frac{1}{2}\mathrm{Tr}\not{k}G^{(B)}_{\lambda\rho}(0,0) &= \frac{g^2}{2M_W^4}\int \frac{d^4p}{(2\pi)^3}\delta(p^2)\eta_e(p)\Big\{k^2\left[k\cdot p\,\eta_{\lambda\rho} - 4k_\lambda p_\rho - 4k_\rho p_\lambda + 4p_\lambda p_\rho\right] \\ &\quad + 4k\cdot p\left[2k_\lambda p_\rho + 2k_\rho p_\lambda - p_\lambda p_\rho - k_\lambda k_\rho\right]\Big\}. \end{aligned} \tag{4.35}$$

Using Eq. (4.19) now and carrying out the integral over p^0, from Eq. (4.22) we obtain

$$\begin{aligned} G^{(B)}(0,0) &= \frac{g^2}{M_W^4}\int \frac{d^3p}{(2\pi)^3 2E}\Big\{8k\cdot p\Big[2k\cdot v\,p\cdot v - k\cdot p\Big](f_e + f_{\bar{e}}) \\ &\quad - 4k\cdot p\Big[(p\cdot v)^2 + (k\cdot v)^2\Big](f_e - f_{\bar{e}})\Big\}. \end{aligned} \tag{4.36}$$

This can be written as

$$G^{(B)}(0,0) = \frac{g^2}{M_W^4}\Big\{16(k\cdot v)k^\mu v^\nu I_{\mu\nu} - 8k^\mu k^\nu I_{\mu\nu} - 4k^\mu v^\nu v^\lambda I_{\mu\nu\lambda} - 4(k\cdot v)^2 k^\mu I_\mu\Big\}, \tag{4.37}$$

and using Eq. (4.32)

$$G^{(B)}(0,0) = \frac{g^2}{M_W^4}\left\{-4\omega^3 J_1^{(e)} + \frac{16}{3}\omega^2 J_2^{(e)} - 4\omega J_3^{(e)}\right\}. \tag{4.38}$$

4.3.3 Computation of $G^{(C)}$ and $G^{(D)}$

Proceeding in similar fashion, we get

$$\frac{1}{2}\mathrm{Tr}\not{k}G^{(C)}_{\lambda\rho}(0,0) = \frac{g^2}{2M_W^4}\int \frac{d^4p}{(2\pi)^3}\delta(p^2)\eta_e(p)(p-k)^2[\eta_{\lambda\rho}(k\cdot p) + p_\lambda k_\rho + k_\lambda p_\rho]. \tag{4.39}$$

Using Eq. (4.19) and Eq. (4.22), we can now write

$$\begin{aligned} G^{(C)}(0,0) &= -\frac{g^2}{M_W^4}\int \frac{d^3p}{(2\pi)^3 2E}\;4k\cdot p\Big[p\cdot v\,k\cdot v - k\cdot p\Big](f_e + f_{\bar{e}}) \\ &= -\frac{g^2}{M_W^4}\Big\{4(k\cdot v)k^\mu v^\nu I_{\mu\nu} - 4k^\mu k^\nu I_{\mu\nu}\Big\}, \end{aligned} \tag{4.40}$$

and similarly it can be verified that $G^{(D)}(0,0)$ is given by the same expression. Using Eq. (4.32) then

$$G^{(C)}(0,0) = G^{(D)}(0,0) = \left(\frac{g^2}{M_W^4}\right)\frac{4}{3}\omega^2 J_2^{(e)}. \tag{4.41}$$

5 Neutrino dispersion relation

Collecting the results for the $G^{(X)}(0, \mathcal{Q} \to 0)$ given in Eqs. (4.34), (4.38) and (4.41) and substituting them in Eq. (4.21), we finally obtain

$$B^{(4)}(\omega, K) = \frac{g^2}{M_W^4}\left[-4\omega^3 J_1^{(e)} + \frac{52}{3}\omega^2 J_2^{(e)}\right]\phi^{\rm ext} \tag{5.1}$$

for $\omega = \pm K$, which is the result that must be substituted in Eq. (4.14) for the dispersion relation of the electron neutrino. For $\nu_{\mu,\tau}$, Eq. (4.14) holds but the $B^{(4)}$ term is absent.

The integrals defined in Eq. (4.31) cannot be evaluated in a general way for arbitrary distributions and $m_f \neq 0$, but for $m_e = 0$ they can be performed exactly to yield the following result in terms of the chemical potential and temperature of the electron gas:

$$\begin{aligned} J_1^{(e)} &= \frac{\mu_e^3}{12\pi^2} + \frac{\mu_e T^2}{12}\,, \\ J_2^{(e)} &= \frac{\mu_e^4}{16\pi^2} + \frac{\mu_e^2 T^2}{8} + \frac{7\pi^2 T^4}{240}\,. \end{aligned} \tag{5.2}$$

For our present purposes, it is sufficient to note that, in any case, $J_1^{(f)}$ and $J_2^{(f)}$ are simply related to the number density n_f and energy density ρ_f of the particles in the background medium by the formulas

$$\begin{aligned} J_1^{(f)} &= \frac{1}{4}(n_f - n_{\bar f})\,, \\ J_2^{(f)} &= \frac{1}{4}(\rho_f + \rho_{\bar f})\,. \end{aligned} \tag{5.3}$$

Thus, using Eq. (5.1), the dispersion relation becomes

$$\begin{aligned} \omega_K &= K + 2K\phi^{\rm ext}(1+\gamma) + (1+\phi^{\rm ext})b_{\rm mat} + b_G^{(2)} \\ \bar\omega_K &= K + 2K\phi^{\rm ext}(1+\bar\gamma) + (1+\phi^{\rm ext})\bar b_{\rm mat} - b_G^{(2)}\,, \end{aligned} \tag{5.4}$$

where

$$\begin{aligned} \gamma &= \left(\frac{g^2}{4M_W^4}\right)\left[-2K(n_e - n_{\bar e}) + \frac{26}{3}(\rho_e + \rho_{\bar e})\right] \qquad \text{for } \nu_e \\ \bar\gamma &= \left(\frac{g^2}{4M_W^4}\right)\left[2K(n_e - n_{\bar e}) + \frac{26}{3}(\rho_e + \rho_{\bar e})\right] \qquad \text{for } \bar\nu_e\,, \end{aligned} \tag{5.5}$$

with γ ($\bar\gamma$) being zero for $\nu_{\mu,\tau}$ ($\bar\nu_{\mu,\tau}$).

The other terms in Eq. (5.4) are the following. In Ref. [4], we calculated the contributions to $b_G^{(2)}$ for various kinds of background particles — non-relativistic or ultra-relativistic, non-degenerate or degenerate, arriving at the final formula given in Eq. (4.30) there. For the electrons, we use here the results for the ultra-relativistic case since it is the one that corresponds to the limit $m_e = 0$ that we have adopted in the present paper. Then,

$$b_G^{(2)} = \left(\frac{g^2}{4M_W^2}\right)\phi^{\rm ext} \times \begin{cases} \mathscr{J}_e + \sum_f X_f \mathscr{J}_f & \text{for } \nu_e\,, \\ \sum_f X_f \mathscr{J}_f & \text{for } \nu_\mu, \nu_\tau \end{cases} \tag{5.6}$$

for $f = e, n, p$, where X_f is the vector neutral-current coupling of the fermion, and

$$\begin{aligned} \mathscr{J}_e &= -5(n_e - n_{\bar{e}}) \\ \mathscr{J}_N &= \begin{cases} -\beta m_N n_N & \text{classical nucleon gas} \\ -\dfrac{3n_N}{v_{FN}^2} & \text{degenerate nucleon gas}\,, \end{cases} \end{aligned} \tag{5.7}$$

for $N = n, p$, where v_{FN} stands for the Fermi velocity of the nucleon gas.

The term $b_{\rm mat}$ can be split into two parts. One is the usual Wolfenstein term, which is given by

$$b_{\rm mat}^{(2)} = \left(\frac{g^2}{4M_W^2}\right) \times \begin{cases} (n_e - n_{\bar{e}}) + \sum_f X_f(n_f - n_{\bar{f}}) & \text{for } \nu_e\,, \\ \sum_f X_f(n_e - n_{\bar{e}}) & \text{for } \nu_\mu, \nu_\tau \end{cases} \tag{5.8}$$

for the neutrinos, and $\bar{b}_{\rm mat}^{(2)} = -b_{\rm mat}^{(2)}$ for the antineutrinos. The second part consists of the corrections of $O(1/M_W^4)$. Using the results of Ref. [8] [in particular Eq. (3.27)], but putting $m_e \to 0$, and neglecting the terms that arise if there are neutrinos and other charged leptons in the background, we can write this part in the form

$$b_{\rm mat}^{(4)} = \left(\frac{g^2}{4M_W^4}\right) \times \begin{cases} -\frac{8}{3}K(\rho_e + \rho_{\bar{e}}) & \text{for } \nu_e\,, \\ 0 & \text{for } \nu_\mu, \nu_\tau \end{cases} \tag{5.9}$$

with $\bar{b}_{\rm mat}^{(4)} = b_{\rm mat}^{(4)}$ for the antineutrinos.

6 Discussions and Conclusions

As the formulas given in Eqs. (5.4) and (5.5) show, the effects of the induced gravitational terms on the neutrino dispersion relation are more noticeable at high energies. An important point to

remember is that the calculations that we have presented rely on the approximation made in Eq. (2) for the W-boson propagator. This implies that the results are valid as long as $(p-k)^2 \ll M_W^2$, which translates to $K\langle E_e\rangle \ll M_W^2$, where $\langle E_e\rangle$ denotes a typical average energy of the electrons in the background. For backgrounds with a temperature of the order of the electron mass or somewhat above, this implies that $K \ll 10^7 GeV$. Thus, if we consider values of $K \approx 10^5 - 10^6 GeV$, for which our results hold, we find that the induced gravitational effects can be substantial if the gravitational potential is not too small. In fact, from Eq. (5.5), it follows in that case that

$$2K\gamma\phi^{\rm ext} \approx -\left(\frac{g^2}{M_W^2}\right)(n_e - n_{\bar{e}})\left(\frac{K}{10^2 GeV}\right)^2 \phi^{\rm ext} \tag{6.1}$$

for ν_e, with $\bar{\gamma} = -\gamma$ for $\bar{\nu}_e$, and zero for the other neutrino and antineutrino types. Moreover, it is possible that, for a certain range of values of the neutrino momentum, the term in Eq. (6.1) is the dominant one in the neutrino dispersion relation. These considerations may be applicable, for example, in the environments of Active Galactic Nuclei. It has been observed that in such systems, high energy neutrinos can have resonant spin-flavor transitions due to the combined effects of the gravitational interactions and the presence of large magnetic fields[9]. Since in such environments $\phi^{\rm ext} \approx 1/10 - 1/100$ in the region in which the neutrinos are produced, the momentum-dependent gravitational terms modify the neutrino dispersion relation in a substantial way, and therefore they can have observable consequences in the determination of the high energy neutrino fluxes from such systems. The calculations that we have presented, and our result for the neutrino dispersion relation, are useful in this and possibly other contexts, and provide a firm basis to study their possible effects in detail.

Appendices

A Proof of Eq. (4.18)

To prove the statement that $(2v^\lambda v^\rho - \eta^{\lambda\rho})Y_{\lambda\rho}(0, \mathcal{Q} \to 0)$ is well defined, we consider the integral formula for $Y_{\lambda\rho}(\Omega, \mathcal{Q})$ given in Eq. (4.16). Carrying out the integration over p^0, in the rest frame of medium, and using the fact that, considered as a function of p and q,

$$j_{\lambda\rho}(-p, q) = -j_{\lambda\rho}(p, -q)\,, \tag{A.1}$$

we have

$$Y_{\lambda\rho}(\Omega, \mathcal{Q}) \equiv \int \frac{d^3P}{(2\pi)^3 2\mathcal{E}}(f_e - f_{\bar{e}})\left[\frac{j_{\lambda\rho}}{q^2 - 2p\cdot q} + (q \to -q)\right]\,, \tag{A.2}$$

where now

$$p^\mu = (\mathcal{E}, \vec{\mathcal{P}}) \qquad \mathcal{E} = |\vec{\mathcal{P}}|\,. \tag{A.3}$$

We consider separately the functions

$$\begin{aligned} Y_\eta &= \eta^{\lambda\rho} Y_{\lambda\rho} \\ Y_{vv} &= v^\lambda v^\rho Y_{\lambda\rho} \,, \end{aligned} \tag{A.4}$$

and show explicitly that each one has a well defined value in the limit ($\Omega = 0, \vec{\mathcal{Q}} \to 0$), which proves the statement.

From Eq. (4.17) we obtain

$$\eta^{\lambda\rho} j_{\lambda\rho} = -(k \cdot p)\frac{1}{2}(2p \cdot q - q^2) \tag{A.5}$$

which implies that

$$\begin{aligned} Y_\eta &= -\int \frac{d^3P}{(2\pi)^3 2\mathcal{E}}(f_e - f_{\bar{e}}) k \cdot p \\ &= \frac{1}{4}(n_e - n_{\bar{e}}) k \cdot v \,. \end{aligned} \tag{A.6}$$

For Y_{vv}, setting $\Omega = 0$ we obtain

$$\begin{aligned} Y_{vv}(0, \vec{\mathcal{Q}}) &= \frac{1}{4}\int \frac{d^3P}{(2\pi)^3}(f_e - f_{\bar{e}})(2\mathcal{E}^2\omega - 2\mathcal{E}\vec{\mathcal{P}} \cdot \vec{K}) \left[\frac{1}{2\vec{\mathcal{P}} \cdot \vec{\mathcal{Q}} - \mathcal{Q}^2} - \frac{1}{2\vec{\mathcal{P}} \cdot \vec{\mathcal{Q}} + \mathcal{Q}^2}\right] \\ &\quad + \frac{1}{4}\int \frac{d^3\mathcal{P}}{(2\pi)^3}(f_e - f_{\bar{e}})(\mathcal{E}\vec{\mathcal{Q}} \cdot \vec{K} - \omega\vec{\mathcal{P}} \cdot \vec{\mathcal{Q}}) \left[\frac{1}{2\vec{\mathcal{P}} \cdot \vec{\mathcal{Q}} - \mathcal{Q}^2} + \frac{1}{2\vec{\mathcal{P}} \cdot \vec{\mathcal{Q}} + \mathcal{Q}^2}\right] . \end{aligned} \tag{A.7}$$

The integrals with the factors $2\mathcal{E}\vec{\mathcal{P}} \cdot \vec{K}$ and $\mathcal{E}\vec{\mathcal{Q}} \cdot \vec{K}$ are zero by symmetric integration since the corresponding integrands are odd under $\vec{\mathcal{P}} \to -\vec{\mathcal{P}}$. The remaining two integrals are evaluated by making the change of variables $\vec{\mathcal{P}} \to \vec{\mathcal{P}} \pm \frac{1}{2}\vec{\mathcal{Q}}$ when the denominator is $2\vec{\mathcal{P}} \cdot \vec{\mathcal{Q}} \mp \mathcal{Q}^2$, respectively, and then taking the limit $\vec{\mathcal{Q}} \to 0$. In this way,

$$\begin{aligned} Y_{vv}(0, \vec{\mathcal{Q}} \to 0) &= \frac{1}{4}\int \frac{d^3\mathcal{P}}{(2\pi)^3 2\mathcal{E}} \frac{d}{d\mathcal{E}}[2\omega\mathcal{E}^2(f_e - f_{\bar{e}})] - \frac{1}{4}\int \frac{d^3\mathcal{P}}{(2\pi)^3}[\omega(f_e - f_{\bar{e}})] \\ &= \frac{1}{4}\int \frac{d^3\mathcal{P}}{(2\pi)^3}\omega\frac{d}{d\mathcal{E}}[\mathcal{E}(f_e - f_{\bar{e}})] \,. \end{aligned} \tag{A.8}$$

B The limit $G^{(A)}(\Omega = 0, \vec{\mathcal{Q}} \to 0)$

The expression for $G^{(A)}$ appears in Eq. (4.26). In the text, we have evaluated the terms without any denominator. Let us denote the remaining terms by $\widehat{G}^{(A)}$. For these terms, we first put $\Omega = 0$. The resulting expression for $\mathscr{M}(p, q)$ is written as

$$\mathscr{M}_0(p, \vec{\mathcal{Q}}) = -4\mathcal{E}^3\omega + 4\mathcal{E}^2\vec{\mathcal{P}} \cdot \vec{K} + 2\mathcal{E}\omega\vec{\mathcal{P}} \cdot \vec{\mathcal{Q}} - 2\mathcal{E}^2\vec{K} \cdot \vec{\mathcal{Q}} \,, \tag{B.1}$$

which follows directly from the definition in Eq. (4.27). We can then write these potentially singular terms in the form

$$
\begin{aligned}
\widehat{G}^{(A)}(0,\vec{\mathcal{Q}}) &= -\frac{g^2}{M_W^4}\int\frac{d^3\mathcal{P}}{(2\pi)^3 2\mathcal{E}}\Bigg[\frac{(k^2-2k\cdot p)\mathscr{M}_0(p,\vec{\mathcal{Q}})}{2\vec{\mathcal{P}}\cdot\vec{\mathcal{Q}}-\mathcal{Q}^2}\,f_e(\mathcal{E})-\frac{(k^2+2k\cdot p)\mathscr{M}_0(-p,\vec{\mathcal{Q}})}{2\vec{\mathcal{P}}\cdot\vec{\mathcal{Q}}+\mathcal{Q}^2}f_{\bar{e}}(\mathcal{E}) \\
&\qquad -\frac{(k'^2-2k'\cdot p)\mathscr{M}_0(p,-\vec{\mathcal{Q}})}{2\vec{\mathcal{P}}\cdot\vec{\mathcal{Q}}+\mathcal{Q}^2}\,f_e(\mathcal{E})+\frac{(k'^2+2k'\cdot p)\mathscr{M}_0(-p,-\vec{\mathcal{Q}})}{2\vec{\mathcal{P}}\cdot\vec{\mathcal{Q}}-\mathcal{Q}^2}f_{\bar{e}}(\mathcal{E})\Bigg] \\
&= -\frac{g^2}{M_W^4}\Big[\widehat{G}_1^{(A)}(0,\vec{\mathcal{Q}})+\widehat{G}_2^{(A)}(0,\vec{\mathcal{Q}})\Big]\,, \qquad \text{(B.2)}
\end{aligned}
$$

where $\widehat{G}_1$ comes from the first pair of terms and $\widehat{G}_2$ from the second pair.

The integrals in this equation can be conveniently expressed in terms of the following ones:

$$
\mathcal{F}_{n,m} = \int\frac{d^3\mathcal{P}}{(2\pi)^3 2\mathcal{E}}\,\mathcal{E}^n(\vec{\mathcal{P}}\cdot\vec{\mathcal{Q}})^m\left[\frac{f_e(\mathcal{E})}{2\vec{\mathcal{P}}\cdot\vec{\mathcal{Q}}-\mathcal{Q}^2}-(-1)^{n+m}\frac{f_{\bar{e}}(\mathcal{E})}{2\vec{\mathcal{P}}\cdot\vec{\mathcal{Q}}+\mathcal{Q}^2}\right], \qquad \text{(B.3)}
$$

as well as

$$
\mathcal{G}^{(n)}_{i_1 i_2\cdots i_r} = \int\frac{d^3\mathcal{P}}{(2\pi)^3 2\mathcal{E}}\mathcal{E}^n\mathcal{P}_{i_1}\mathcal{P}_{i_2}\cdots\mathcal{P}_{i_r}\left[\frac{f_e(\mathcal{E})}{2\vec{\mathcal{P}}\cdot\vec{\mathcal{Q}}-\mathcal{Q}^2}-(-1)^{n+r}\frac{f_{\bar{e}}(\mathcal{E})}{2\vec{\mathcal{P}}\cdot\vec{\mathcal{Q}}+\mathcal{Q}^2}\right], \qquad \text{(B.4)}
$$

where i_1 etc are spatial indices. Using these notations, we can write the first pair of terms for $\widehat{G}^{(A)}$ as

$$
\begin{aligned}
\widehat{G}_1^{(A)}(0,\vec{\mathcal{Q}}) &= 16\omega^2\mathcal{F}_{4,0}+8\omega\vec{K}\cdot\vec{\mathcal{Q}}\mathcal{F}_{3,0}-8\omega^2\mathcal{F}_{2,1} \\
&\quad -32\omega K_i\mathcal{G}_i^{(3)}-8\vec{K}\cdot\vec{\mathcal{Q}}\,K_i\mathcal{G}_i^{(2)}+16K_iK_j\mathcal{G}_{ij}^{(2)}+8\omega\mathcal{Q}_iK_j\mathcal{G}_{ij}^{(1)}\,, \qquad \text{(B.5)}
\end{aligned}
$$

where we have now used Eq. (4.19). To simplify further, we first notice that by elementary arguments the integrals in Eq. (B.4) can be expressed in terms of the integrals $\mathcal{F}_{n,m}$:

$$
\begin{aligned}
\mathcal{G}_i^{(n)} &= \frac{\mathcal{Q}_i}{\mathcal{Q}^2}F_{n,1} \\
\mathcal{G}_{ij}^{(n)} &= \frac{1}{2}\left[\left(F_{n+2,0}-\frac{F_{n,2}}{\mathcal{Q}^2}\right)\delta_{ij}-\left(F_{n+2,0}-\frac{3F_{n,2}}{\mathcal{Q}^2}\right)\frac{\mathcal{Q}_i\mathcal{Q}_j}{\mathcal{Q}^2}\right]. \qquad \text{(B.6)}
\end{aligned}
$$

The next step is to realize that the scalar integrals $\mathcal{F}_{n,m}$ for a particular value of m can be expressed in terms of $\mathcal{F}_{n,m}$ for smaller values of m and the integrals $J_n^{(e)}$ defined in Eq. (4.31). For example,

$$
\begin{aligned}
\mathcal{F}_{n,1} &= \frac{1}{2}\Big[J_n^{(e)}+\mathcal{Q}^2\mathcal{F}_{n,0}\Big]\,, \\
\mathcal{F}_{n,2} &= \frac{1}{2}\mathcal{Q}^2\mathcal{F}_{n,1}\,, \qquad \text{(B.7)}
\end{aligned}
$$

Finally, in $\mathcal{F}_{n,0}$, the angular integration can be performed exactly to give

$$\mathcal{F}_{n,0} = \frac{1}{16\pi^2} \int_0^\infty d\mathcal{E}\, \frac{\mathcal{E}^n}{\mathcal{Q}}\, \ln\left(\frac{1-(\mathcal{Q}/2\mathcal{E})}{1+(\mathcal{Q}/2\mathcal{E})}\right) \times \left[f_e(\mathcal{E}) + (-1)^n f_{\bar{e}}(\mathcal{E})\right]. \tag{B.8}$$

For small $\mathcal{Q}$, the logarithm can be expanded in power series and we obtain

$$\mathcal{F}_{n,0} \;=\; -\frac{1}{4}\, J^{(e)}_{n-2} - \frac{\mathcal{Q}^2}{48}\, J^{(e)}_{n-4} + O\left(\mathcal{Q}^4\right). \tag{B.9}$$

The integrals for larger values of m can be obtained by using Eq. (B.7).

When we put the results of these integrals, Eq. (B.5) reduces to the form

$$\widehat{G}^{(A)}_1(0,\vec{\mathcal{Q}}) \;=\; -12\omega^2 J^{(e)}_2 + \frac{4(\vec{K}\cdot\vec{\mathcal{Q}})^2}{\mathcal{Q}^2} J^{(e)}_2 - \frac{16\omega(\vec{K}\cdot\vec{\mathcal{Q}})}{\mathcal{Q}^2} J^{(e)}_3 + \mathcal{O}(\mathcal{Q}^2)\,, \tag{B.10}$$

In a similar fashion, the other pair of terms in $\widehat{G}^{(A)}$ can be evaluated. The result is

$$\widehat{G}^{(A)}_2(0,\vec{\mathcal{Q}}) \;=\; -12\omega^2 J^{(e)}_2 - \frac{4(\vec{K}\cdot\vec{\mathcal{Q}})^2}{\mathcal{Q}^2} J^{(e)}_2 + \frac{16\omega(\vec{K}\cdot\vec{\mathcal{Q}})}{\mathcal{Q}^2} J^{(e)}_3 - 8\omega J^{(e)}_3 + \mathcal{O}(\mathcal{Q}^2)\,. \tag{B.11}$$

Summing these two contributions and reinstating the overall factor, we can write

$$\widehat{G}^{(A)}(0,0) = \frac{g^2}{M_W^4}\left[12\omega^2 J^{(e)}_2 + 4\omega J^{(e)}_3\right]. \tag{B.12}$$

Adding this with the contribution of the regular terms obtained in Eq. (4.33), we get the total result shown in Eq. (4.34).

An alternative evaluation of Eq. (B.12) is as follows. In the $\mathcal{Q}$-dependent integrals we make the change of variables $\vec{\mathcal{P}} \to \vec{\mathcal{P}} \pm \frac{1}{2}\vec{\mathcal{Q}}$ in those terms containing in the denominator the factors $\vec{\mathcal{P}}\cdot\vec{\mathcal{Q}} \mp \mathcal{Q}^2$, respectively. By expanding the integrands in powers of $\mathcal{Q}$ and taking the limit $\mathcal{Q}\to 0$ we obtain in this fashion

$$\begin{aligned}
\widehat{G}^{(A)}(0,\vec{\mathcal{Q}}\to 0) \;&=\; -\frac{g^2}{M_W^4}\int \frac{d^3\mathcal{P}}{(2\pi)^3 2\mathcal{E}}\, 2k\cdot p\Bigg\{\left[|\vec{\mathcal{P}}-\vec{K}|^2 \frac{d}{d\mathcal{E}}\left(\mathcal{E} f_e(\mathcal{E})\right) - \frac{d}{d\mathcal{E}}\left[(\mathcal{E}-\omega)^2 \mathcal{E} f_e\right]\right] \\
&\qquad -\left[|\vec{\mathcal{P}}+\vec{K}|^2 \frac{d}{d\mathcal{E}}\left(\mathcal{E} f_{\bar{e}}(\mathcal{E})\right) - \frac{d}{d\mathcal{E}}\left[(\mathcal{E}+\omega)^2 \mathcal{E} f_{\bar{e}}\right]\right]\Bigg\} \\
&=\; \frac{g^2}{M_W^4}\int \frac{d^3\mathcal{P}}{(2\pi)^3 2\mathcal{E}}\, 4k\cdot p\Bigg\{\left[-k\cdot p \frac{d(\mathcal{E} f_e)}{d\mathcal{E}} + \mathcal{E}(\mathcal{E}-\omega) f_e\right] \\
&\qquad -\left[k\cdot p \frac{d(\mathcal{E} f_{\bar{e}})}{d\mathcal{E}} + \mathcal{E}(\mathcal{E}+\omega) f_{\bar{e}}\right]\Bigg\}\,,
\end{aligned} \tag{B.13}$$

using Eq. (4.19) in the last step. Carrying out an integration by parts and combining some terms, we get

$$\widehat{G}^{(A)}(0,\vec{\mathcal{Q}}\to 0) \quad = \quad \frac{g^2}{M_W^4}\int \frac{d^3\mathcal{P}}{(2\pi)^3 2\mathcal{E}}\left[12(k\cdot p)^2(f_e+f_{\bar{e}})+4k\cdot p(\mathcal{E}^2-\mathcal{E}\omega)(f_e-f_{\bar{e}})\right], \quad \text{(B.14)}$$

which can be written in the form

$$G^{(A)}(0,\vec{\mathcal{Q}}\to 0) = \frac{g^2}{M_W^4}\left[12k^\mu k^\nu I_{\mu\nu}+4k^\mu v^\nu v^\lambda I_{\mu\nu\lambda}-4\omega k^\mu v^\nu I_{\mu\nu}\right], \quad \text{(B.15)}$$

in terms of the integrals defined in Eq. (4.29). Substituting into Eq. (B.15) the formulas given in Eq. (4.32), the result given in Eq. (B.12) is reproduced.

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